Now, using (7.4), we can write

$$\mathbf{a}_{\mathrm{tr}}|_{\mathfrak{e}_\alpha=\mathrm{const}} = \mathbf{a}_{\mathrm{tr}}|_{\hat{\mathfrak{e}}_\alpha=\mathrm{const}} + \frac{\partial v_{\mathrm{tr}}}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial \tau}\Big|_{\hat{\mathfrak{e}}_\alpha=\mathrm{const}} + v^\alpha_{\mathrm{rel}}(\nabla_\alpha v_{\mathrm{tr}\,\beta})\,\mathfrak{e}^\beta. \tag{7.6}$$

Differentiating (7.2) with respect to τ from the viewpoint of an observer with basis $\mathfrak{e}_\alpha$ (assumed to be inertial), we find

$$\mathbf{a}_{\mathrm{abs}} = \mathbf{a}_{\mathrm{tr}}|_{\hat{\mathfrak{e}}_\alpha=\mathrm{const}} + \mathbf{a}_{\mathrm{rel}}|_{\hat{\mathfrak{e}}_\alpha=\mathrm{const}} + 2v^\alpha_{\mathrm{rel}}\nabla_\alpha v_{\mathrm{tr}\,\beta}\mathfrak{e}^\beta \tag{7.7}$$

Formula (7.7) is a generalization of the classical Coriolis formula, which is established to Newtonian mechanics and derived in Cartesian coordinate systems with orthonormal bases $\mathfrak{e}_\alpha$ and $\hat{\mathfrak{e}}_\alpha$.

Formula (7.7) holds in both special and general relativity theory, in any curvilinear coordinates.

If one uses the equality

$$\nabla_\alpha v_{\mathrm{tr}\,\beta} = {}^1/_2(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha) + {}^1/_2(\nabla_\alpha v_\beta - \nabla_\beta v_\alpha) = e_{\alpha\beta} + \omega_{\alpha\beta}$$

where $e_{\alpha\beta}$ are the components of the rate of strain and $\omega_{\alpha\beta}$ those of the rotation tensor in translational motion in the volume Σ, the generalized Coriolis formula (7.7) may be written as

$$\mathbf{a}_{\mathrm{abs}} = \mathbf{a}_{\mathrm{tr}} + \mathbf{a}_{\mathrm{rel}} + 2v^\alpha_{\mathrm{rel}}(e_{\alpha\beta} + \omega_{\alpha\beta})\,\mathfrak{e}^\beta \tag{7.8}$$

where each term is represented in a tangible form.

Formula (7.8) retains its form when the accelerations are considered relative to a non-inertial observer with basis $\mathfrak{e}_\alpha$.

Though based on the most elementary concepts of tensor analysis, the foregoing arguments provide a more general result in a more general situation, with practically no computations, at the same time demonstrating the reason for the appearance and nature of the "added" acceleration in formula (7.8).

The motion of a moving point M is split up into absolute and relative motions owing to the introduction of reference frames of translational motion; such frames may be introduced holonomically, together with global time, or locally - and in general non-holonomically - for each position of the moving point M.

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INVERSION OF LAGRANGE'S THEOREM FOR A RIGID BODY WITH A CAVITY CONTAINING A VISCOUS LIQUID*

V.A. VLADIMIROV and V.V. RUMYANTSEV

The stability of the state of equilibrium of a rigid body with a cavity partly or completely filled with a viscous incompressible liquid possessing surface tension is cosidered in a linear form. Lyapunov's direct method is used to show that the system is unstable if the second variation of the potential energy can take negative values. A priori lower and upper bounds for the solutions, when the perturbations are increased, are obtained. The lower bound guarantees exponential growth of the deviations of the solid and liquid particles from the equilibrium state. The upper bound shows that the solutions cannot increase at more than an exponential rate. In both cases the exponents are calculated from the parameters of the equilibrium state and the initial data for the perturbation fields.

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1. General description of the system. We are interested in the motion of a rigid body with a cavity containing a liquid. We introduce a fixed (inertial) system of Cartesian coordinates $O'x_1'x_2'x_3'$ and a moving system $Ox_1x_2x_3$ rigidly attached to the rigid body, with its origin at a certain point O of the body. In $Ox_1x_2x_3$ coordinates the body and the cavity occupy fixed regions τ_1 and τ. The boundary $\partial\tau$ is the outer boundary of the cavity (τ) and at the same time the inner boundary of the body (τ_1). The density of the body is given by a function $\rho = \rho(x_1, x_2, x_3)$. The region τ is completely filled with two liquids. At any instant of time a liquid surface Γ divides τ into two parts τ^+ and τ^-, containing liquids with coefficients of viscosity η^+ and η^- and densities ρ^+ and ρ^-. In $\tau^\pm$ the quantities $\rho^\pm$, $\eta^\pm$ are constant, but across Γ there occur jumps of density $[\rho] \equiv \rho^+ - \rho^-$ and viscosity $[\eta] \equiv \eta^+ - \eta^-$.

The curve γ in which the surfaces $\partial\tau$ and Γ intersect divides $\partial\tau$ into parts $\partial\tau^+$ and $\partial\tau^-$. On the surfaces Γ, $\partial\tau^+$, $\partial\tau^-$ the respective liquids have constant surface tensions σ, σ^+, σ^-. The symbols ν and $\mathbf{n}$ denote unit normals to the surfaces Γ and $\partial\tau$, ν being directed from the liquid ρ^+ to ρ^-, and $\mathbf{n}$ from τ to τ_1.

It is assumed that the rigid body is either constrained by certain stationary geometrical constraints or is free. The number of degrees of freedom is denoted by n ($n \leqslant 6$). The position of the system is given by the generalized coordinates of the body q_α ($\alpha = 1, \ldots, n$) and the relative coordinates of the liquid particles x_i ($i = 1, 2, 3$).

As indices we shall use letters of the Greek and Latin alphabets. The former take values from 1 to n and correspond to finitely indexed degrees of freedom. The latter vary from 1 to 3 and denote components of vectors and tensors. Throughout we shall employ the summation convention with respect to repeated indices (both Latin and Greek). The following abbreviated notation is adopted for functions:

$$\varphi(x_k, q_\alpha) \equiv \varphi(x_1, x_2, x_3, q_1, \ldots, q_n)$$

The position of a material particle of the body or the liquid relative to the fixed system of coordinates $O'x_1'x_2'x_3'$ and moving system of coordinates $Ox_1x_2x_3$ is given by the vectors $\mathbf{r}'$ and $\mathbf{r}$, respectively. With this notation we have

$$\mathbf{r}' = \mathbf{r} + \mathbf{r}_0' = \mathbf{r}'(q_\alpha, x_i) \tag{1.1}$$

where $\mathbf{r}_0'$ is the radius-vector of the point O in the system $Ox_1'x_2'x_3'$. Differentiating (1.1) with respect to time, we obtain

$$\mathbf{v} = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{u} \tag{1.2}$$

where $\mathbf{v}_0$ is the velocity vector of the point O of the body, $\boldsymbol{\omega}$ is the instantaneous angular velocity vector, and $\mathbf{v}$ and $\mathbf{u}$ are the velocities of the material point relative to the coordinate systems $O'x_1'x_2'x_3'$ and $Ox_1x_2x_3$. For points of the rigid body $\mathbf{u} \equiv 0$. We have

$$\mathbf{r} = x_i\mathbf{j}_i, \quad \mathbf{u} = u_i\mathbf{j}_i, \quad \mathbf{v} = v_i\mathbf{j}_i \tag{1.3}$$

where $\mathbf{j}_i$ are unit vectors along the axes Ox_i.

When the body and the liquid are in motion, small increments to $\mathbf{r}'$, x_i and q_α are related by the equation

$$\Delta\mathbf{r}' = \frac{\partial\mathbf{r}'}{\partial x_i}\Delta x_i + \frac{\partial\mathbf{r}'}{\partial q_\alpha}\Delta q_\alpha \tag{1.4}$$

and the corresponding relation for the velocities is

$$\mathbf{v} = \frac{\partial\mathbf{r}'}{\partial x_i}u_i + \frac{\partial\mathbf{r}'}{\partial q_\alpha}\dot{q}_\alpha, \quad \dot{q}_\alpha \equiv \frac{dq_\alpha}{dt} \tag{1.5}$$

Forces exterior to the "body + liquid" system are applied both to the body and to the liquid. The forces acting on the body are characterized by their potential energy $\Pi_1 = \Pi_1(q_\alpha)$ and dissipation function $R_n(q_\alpha, \dot{q}_\beta)$, where the dissipation may be either complete or partial. Acting on the liquid is an external field of body forces with potential $\Phi(\mathbf{r}')$. Substituting (1.1) into the potential of external forces applied to the liquid, we obtain

$$\Phi(\mathbf{r}'(q_\alpha, x_i)) \equiv \Pi_2(q_\alpha, x_i) \tag{1.6}$$

The potential energy of the system is given by

$$\Pi_n = \Pi_1 + \int_\tau \rho\Pi_2 d\tau + \Pi_2^* \tag{1.7}$$

$$\Pi_2^* \equiv \sigma|\Gamma| + \sigma^+|\partial\tau^+| + \sigma^-|\partial\tau^-|$$

where $|\Gamma|$, $|\partial\tau^{\pm}|$ are the areas of the appropriate surfaces; the potential energy of the surface tension forces Π_2^* is independent of q_α.

The kinetic energy of the system is expressed as

$$T_n = {}^1/_2\int_\Omega \rho v^2 d\tau = T_1 + T_2 \quad \Omega = \tau_1 \cup \tau, \tag{1.8}$$

$$T_1 = T_1(q_\alpha, q_\beta^{\cdot}), \quad T_2 = \int_\tau \rho T_0 d\tau, \qquad T^\circ = {}^1/_2 v^2 = T^\circ(q_\alpha, q_\beta^{\cdot}, x_i, u_k)$$

where T_1 and ρT° are the kinetic energy of the body and the density of the kinetic energy of the liquid; the index n attached to Π_n and T_n corresponds to energy values in the nonlinear problem; by integration over the region Ω we mean the sum of integrals over the regions τ_1, τ^+, τ^-; the coefficient ρ in (1.7), (1.8) for each of these integrals takes the value of the appropriate density: $\rho = \rho_1(x_i)$ in τ_1, $\rho = \rho^{\pm}$ in $\tau^{\pm}$.

2. The equations of motion of the system. When there are no dissipative effects, the equations of motion of the system are those derived in /1, 2/. Allowance for the viscosity of the liquid and the dissipative nature of the external forces applied to the body leads to the equations

$$\frac{d}{dt}\frac{\partial T_n}{\partial q_\alpha^{\cdot}} - \frac{\partial T_n}{\partial q_\alpha} = -\frac{\partial \Pi_n}{\partial q_\alpha} - \frac{\partial R_n}{\partial q_\alpha^{\cdot}} \tag{2.1}$$

$$\rho\frac{dv_i}{dt} + e_{ikl}\omega_k v_l = -\frac{\partial \Pi_2}{\partial x_i} + \frac{\partial \sigma_{ik}}{\partial x_k}, \quad \frac{\partial u_i}{\partial x_i} = 0 \text{ в } \tau^{\pm} \tag{2.2}$$

$$\sigma_{ik} \equiv -p\delta_{ik} + \eta D_{ik}, \quad D_{ik} \equiv \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}, \quad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + u_k\frac{\partial}{\partial x_k}$$

in which $\mathbf{v}, \boldsymbol{\omega}$ and Π_2 are as in (1.2), (1.6), p is the pressure and e_{ikl} is the absolutely antisymmetric unit tensor. The rank 3 (pseudo-) tensor e_{ikl} is the totality of quantities with the following properties: interchanging any two indices changes the sign of e_{ikl}; $e_{123} = 1$.

The boundary conditions for the liquid are as follows:

$$dF/dt = 0, \quad [\sigma_{ik}]\nu_k = 2\sigma H\nu_i, \quad [u_i] = 0 \quad \text{on } \Gamma \tag{2.3}$$

$$2H \equiv R_1^{-1} + R_2^{-1}$$

$$\mathbf{u} = 0 \text{ on } \partial\tau; \quad \partial F/\partial t = 0 \text{ on } \gamma \tag{2.4}$$

where $F(x_1, x_2, x_3, t) = 0$ is the equation of the surface Γ; H, R_1, R_2 are the mean curvature of Γ and its principal radii of curvature at the point in question. The quantities R_i and R_2 are assumed to be positive if the centre of curvature lies on the same side of the surface as the liquid ρ^+, negative otherwise. The curve γ is the intersection of the surfaces $\partial\tau$ and Γ.

The boundary conditions on the solid surface $\partial\tau$ for a viscous capillary liquid have not yet been established uniquely, and the matter is still controversial /3-8/*. (*See also Voinov O.V., Hydrodynamical Theory of Wetting. Preprint 179, Novosibirsk, Inst. Teplofiziki Sib. Otd. Akad. Nauk SSSR, 1988.) Formula (2.4) corresponds to the simplest version of the boundary conditions used in the literature /7/. More complicated formulations will be discussed in Sect.11.

The solutions of problem (2.1)-(2.4) satisfy the energy equality:

$$E_n^{\cdot} = -D_n - q_\alpha^{\cdot}\partial R_n/\partial q_\alpha^{\cdot}; \quad E_n = T_n + \Pi_n, \tag{2.5}$$

$$2D_n = \int_\tau \eta D_{ik}D_{ik}d\tau$$

The quantities T_n and Π_n are defined in (1.7) and (1.8).

We shall assume that all quantities in (2.1)-(2.5) and below have been reduced to nondimensional form by suitable scaling.

3. Equilibrium states and the second variation of the potential energy. The stationary points of the functional Π_n (1.7) correspond to the equilibrium (rest) states of the system:

$$q_\alpha = 0, \quad q_\beta^{\cdot} = 0$$

$$\mathbf{u} = 0, \quad p = p_0(x_k); \quad \nabla p_0 = -\rho\nabla\Phi \text{ in } \tau_0^{\pm} \tag{3.1}$$

$$[p_0] = -2\sigma H, \quad [\rho] \neq 0 \text{ on } \Gamma$$

where Γ_0 is the equilibrium surface of the density jump, which divides the region τ into parts $\tau_0^{\pm}$. Eqs.(3.1) determine the exact solutions of problem (2.1)-(2.4).

The first and second variation of the potential energy Π_n (1.7) depend only on the geometric parameters of the system. The first variation $\delta\Pi_n$, evaluated in the rest state (3.1), is zero. Calculations of the second variation lead to the expression /2/

$$\delta^2\Pi_n = \frac{1}{2}\left(\frac{\partial^2\Pi_n}{\partial q_\alpha \partial q_\beta}\right)_0 \delta q_\alpha \delta q_\beta + \delta q_\alpha [\rho]\int_{\Gamma_0}\left(\frac{\partial \Pi_2}{\partial q_\alpha}\right)_0 \delta N dS + \tag{3.2}$$
$$\frac{[\rho]}{2}\int_{\Gamma_0}\left(\frac{\partial \Pi_2}{\partial \nu}\right)_0 (\delta N)^2 dS + \frac{\sigma}{2}\int_{\Gamma_0}\{a(\delta N)^2 + \nabla(\delta N, \delta N)\} dS + \frac{\sigma}{2}\int_{\gamma_0}\chi(\delta N)^2 dl$$

$$\delta N \equiv \delta \mathbf{r}\cdot\boldsymbol{\nu}, \quad \partial\Pi_2/\partial\nu \equiv (\boldsymbol{\nu}\nabla)\,\Pi_2$$
$$a \equiv R_1^{-2} + R_2^{-2}, \quad \chi \equiv (k\cos\theta - \bar{k})/\sin\theta$$

in which $\delta\mathbf{r}$ and δq_α are the virtual increments of the radius-vector of relative coordinates of the liquid particles and the generalized coordinate q_α; the zero index corresponds throughout to the values of the functions at equilibrium, γ_0 is the intersection of the surfaces Γ_0 and $\partial\tau$, θ is the contact angle, k and $\bar{k}$ are the curvatures of the normal sections of the surfaces Γ_0 and $\partial\tau$ along the directions $\mathbf{e}$ and $\mathbf{e}_1$, the vectors $\mathbf{e}$ and $\mathbf{e}_1$ in turn are normal to the curve γ and lie in planes tangent to the surfaces Γ_0 (directed out of Γ_0) and $\partial\tau$ (directed out of $\partial\tau_0^+$), respectively, and $\nabla(\delta N, \delta N)$ is the first Beltrami differential parameter /9/.

4. Formulation of the linearized problem. We present a brief derivation of the linearized version of problem (2.1)-(2.5) in the equilibrium state (3.1).

Linear perturbations of the velocity fields $\mathbf{v}(x_i, t)$, $\mathbf{u}(x_i, t)$ satisfy the following equation, which follows from (1.3) and (1.5):

$$v_i = u_i + a_{i\alpha}\dot{q}_\alpha; \quad a_{i\alpha} \equiv (\partial \mathbf{r}'/\partial q_\alpha)_0 \mathbf{j}_{i0} \tag{4.1}$$

The functions $\mathbf{v}(x_i, t)$ and $\mathbf{u}(x_i, t)$ are defined for $x_i \in \tau_1$ and $x_i \in \tau_0^{\pm}$, with $\mathbf{u} \equiv 0$ for $x_i \in \tau_1$. The expression for the first (quadratic) term in the expansion of the kinetic energy is obtained by substituting (4.1) into the definition (1.8):

$$T = \frac{1}{2}\int_{\Omega_0}\rho v_k v_k d\tau = \frac{1}{2}M_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta + \frac{1}{2}\int_{\tau_0}\rho(u_k u_k + 2a_{k\alpha}u_k\dot{q}_\alpha)\,d\tau \tag{4.2}$$
$$M_{\alpha\beta} \equiv \int_{\Omega_0}\rho a_{k\alpha}a_{k\beta}d\tau, \quad \Omega_0 = \tau_1 \cup \tau_0^+ \cup \tau_0^-$$

To derive the first term in the expansion of the potential energy (1.7), we must consider the field of displacements, i.e., the linear increments to the coordinates of a material point, reckoned from its equilibrium position. Retaining the notation (1.4), we write

$$\Delta\mathbf{r}' = \mathbf{r}' - \mathbf{X}' \equiv \boldsymbol{\xi}, \quad \Delta x_i = x_i - X_i \equiv \zeta_i, \quad \Delta q_\alpha = q_\alpha$$

where $\mathbf{X}', X_i$ are the radius-vectors and coordinates of the material points at equilibrium. As a corollary of (1.4) we have

$$\xi_i = \zeta_i + a_{i\alpha}q_\alpha, \quad \xi_i = \boldsymbol{\xi}\mathbf{j}_{i0} \tag{4.3}$$

in which we have made use of (4.1) and (1.3). The fields of displacements $\xi_i(x_k, t)$ and $\zeta_i(x_k, t)$ are also defined for $x_i \in \tau_1$ and $x_i \in \tau_0^{\pm}$, with $\zeta_i = 0$ for $x_i \in \tau_1$. The fields (4.1) and (4.3) are related by the equations

$$\xi_{it} \equiv \partial\xi_i(x_k, t)/\partial t = v_i(x_k, t), \quad \zeta_{it} = u_i(x_k, t) \tag{4.4}$$

whose simple form is a consequence of the fact that the problem is linearized with the system at equilibrium (3.1) (/10/, Sect.13).

The first term in the expansion of the potential energy Π_n near the rest state (3.1) is known to be the second variation $\delta^2\Pi_n$ (3.2), in which we must substitute $\delta q_\alpha \to q_\alpha$, $\delta x_i \to \zeta_i$, $\delta N \to N$:

$$\Pi = \frac{1}{2}\left(\frac{\partial^2\Pi}{\partial q_\alpha \partial q_\beta}\right)_0 q_\alpha q_\beta + [\rho]q_\alpha\int_{\Gamma_0}\left(\frac{\partial\Pi_2}{\partial q_\alpha}\right)_0 N dS + \tag{4.5}$$
$$\frac{[\rho]}{2}\int_{\Gamma_0}\left(\frac{\partial\Pi_2}{\partial\nu}\right)_0 N^2 dS + \frac{\sigma}{2}\int_{\Gamma_0}\{aN^2 + \nabla(N, N)\}\,dS + \int_{\gamma_0}\chi N^2 dl, \quad N \equiv \zeta_i\nu_{0i}$$

and in our model (2.4), (1.5) the integral over γ_0 in (4.5) vanishes.

Eqs.(2.1) and (2.2), linearized by (4.1), become

$$M_{\alpha\beta}q_\beta^{\cdot\cdot} + \int_{\tau_0}^{\cdot} \rho a_{k\alpha}\zeta_{ktt}d\tau = -\frac{\partial R}{\partial q_\alpha^{\cdot}} - \tag{4.6}$$

$$\left(\frac{\partial^2\Pi_1}{\partial q_\alpha \partial q_\beta}\right)_0 q_\beta - [\rho]\int_{\Gamma_0}\left(\frac{\partial \Pi_2}{\partial q_\alpha}\right)_0 \zeta_k \nu_{0k}dS$$

$$\xi_{itt} = \zeta_{itt} + a_{i\alpha}q_\alpha^{\cdot\cdot} = -\frac{1}{\rho}\frac{\partial \sigma_{ik}}{\partial x_k},\ \frac{\partial \zeta_k}{\partial x_k} = 0 \text{ in } \tau_0^{\pm} \tag{4.7}$$

$$\sigma_{ik} = -p\delta_{ik} + \eta D_{ik},\quad 2R \equiv c_{\alpha\beta}q_\alpha^{\cdot}q_\beta^{\cdot} \tag{4.8}$$

where $p = p(x_k, t)$ is the field of pressure perturbations, R the dissipative function in the linear approximation, which is the first term in the expansion of the complete dissipation function $R_n(q_\alpha, q_\beta^{\cdot})$ (2.1), $c_{\alpha\beta}$ are constants. Eq.(4.6) is obtained by substituting (4.2), (4.5) and (4.8) into (2.1).

Linearization of conditions (2.3) gives

$$[\sigma_{ik}]\nu_k + \sigma(aN - \Delta N)\nu_i = [\rho]\left\{\zeta_k\left(\frac{\partial \Pi_2}{\partial x_k}\right)_0 + q_\alpha\left(\frac{\partial \Pi_2}{\partial q_\alpha}\right)\right\};\quad [\zeta] = 0 \text{ on } \Gamma_0 \tag{4.9}$$

where Δ is the second Beltrami differential parameter /9, p.190/, and the part of (4.9) that contains it is derived by using the formulae for the variation of curvature /9, p.276/.

On $\partial\tau$ the no-slip conditions hold:

$$\zeta = 0 \text{ on } \partial\tau \tag{4.10}$$

The initial data for (4.6)-(4.10) are written as

$$\zeta(x_k, 0) = \zeta^\circ(x_k),\quad \zeta_t(x_k, 0) = \mathbf{u}^\circ(x_k)$$
$$q_\alpha(0) = q_\alpha^\circ,\quad q_\alpha^{\cdot}(0) = q_\alpha^{\cdot\circ} \tag{4.11}$$

where the functions $\zeta^\circ, \mathbf{u}^\circ$ must obey obvious kinematic constraints, and q_α° and $q_\alpha^{\cdot\circ}$ are arbitrary constants. Based on (4.11), (4.1) and (4.3) one evaluates the initial data

$$\xi(x_k, 0) = \xi^\circ(x_k),\quad \mathbf{v}(x_k, 0) = \mathbf{v}^\circ(x_k) \tag{4.12}$$

In solutions of problem (4.6)-(4.12) one has the energy equality:

$$E^{\cdot} = -2R - D;\quad E = T + \Pi,\quad 2D \equiv \int_{\tau_0}\eta D_{ik}D_{ik}d\tau \tag{4.13}$$

where T, Π and R are defined in (4.2), (4.5) and (4.8).

The aim of the subsequent account is to establish certain facts about the growth of perturbations of the (unstable) equilibrium state (3.1), on the assumption that the potential energy functional Π (4.5) has no minimum there. It will be assumed that the equilibrium states in question are such that there exists a set Q of displacements $\{\zeta(x_k), q_\alpha\}$ for which (4.5) gives

$$\Pi < 0 \quad \text{for } \{\zeta(x_k), q_\alpha\} \in Q \tag{4.14}$$

and if $\{\zeta(x_k), q_\alpha\} \notin Q$ inequality (4.14) may be replaced by the reverse inequality, i.e., state (3.1) is an infinite-dimensional analogue of a "saddle" point for the functional Π.

5. ***Basic functionals and the generalized virial.*** To apply Lyapunov's direct method, we define the following functionals M, W, G and X and the function φ :

$$M \equiv \int_{\Omega_1}\rho\xi_i\xi_i d\tau,\quad \frac{M^{\cdot}}{2} = W \equiv \int_{\Omega_1}\rho\xi_i v_i d\tau_1 \tag{5.1}$$

$$2G \equiv \int_{\tau_0}\eta G_{ik}G_{ik}d\tau_1,\quad G_{ik} = \frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i},$$

$$X \equiv M^{\cdot} + 2A,\quad 2A \equiv G + \varphi,\quad \varphi = c_{\alpha\beta}q_\alpha q_\beta,\quad \tau_0 = \tau_0^+ \cup \tau_0^-,\quad \Omega_0 = \tau_1 \cup \tau_0$$

where the relation $M^{\cdot} = 2W$ follows from (4.4), G and φ are by definition non-negative, and $c_{\alpha\beta}$ are defined in (4.8). According to /11/, the functional W we can written as

$$W = \frac{\partial T}{\partial q_\alpha} q_\alpha + \int_{\tau_*} \rho v_i \xi_i d\tau \tag{5.2}$$

Differentiation of X with respect to time and subsequent reduction using (5.1), (5.2), (4.2) and (4.5)-(4.10) yield

$$X^\cdot = 4(T - \Pi) = 8T - 4E \tag{5.3}$$

When there are no dissipative effects, equalities of this type are known as virial equations /10/. We may therefore call (5.3) a generalized virial equation.

6. A linear bound for the growth of perturbations. Under conditions (4.14), the simplest bound for the growth of the solution, which indicates instability, is obtained by integrating (5.3) with $E(0) < 0$. The decrease in $E(t)$ with time (4.13) implies $E(t) < E(0) < 0$. It then follows from (5.3) that

$$X(t) > X(0) + 4 \mid E(0) \mid t$$

Hence, after using the inequality $X \leqslant X_1 \equiv M + 2T + 2A$, we obtain the required bound:

$$X_1(t) > X(0) + 4 \mid E(0) \mid t \tag{6.1}$$

in which we can always choose $X(0) = M^\cdot(0) + 2A(0) > 0$.

Since the functional X_1 is positive, inequality (6.1) guarantees that the perturbations will increase linearly in the mean square. At the same time either the velocities v_i, $q_\alpha^\cdot$, the displacements ξ_i, q_α, the liquid strain tensor G_{ik}, or various sums of these magnitudes will increase. The definition of instability will then involve not only increasing deviations of the liquid particles from their equilibrium positions, but also increasing derivatives of these deviations throughout the volume of the liquid. Below we shall present a bound for the growth of the perturbations which will involve only displacements and velocities.

7. Fundamental inequality. Multiply Eq.(5.3) by an undetermined constant multiplier λ and add the result to the energy relation (4.13). After some simple algebra we obtain

$$\frac{d}{dt}(T_\lambda + \Pi_\lambda) = -2\lambda(T_\lambda - \Pi_\lambda) - D_\lambda - R_\lambda \tag{7.1}$$

$$2T_\lambda \equiv 2T - \lambda M^\cdot + \lambda^2 M = \int_{\Omega_*} \rho(\xi_t - \lambda\xi)^2 d\tau, \quad 2\Pi_\lambda \equiv 2\Pi + 2A\lambda + \lambda^2 M$$

$$D_\lambda \equiv D - \lambda G^\cdot + \lambda^2 G = \frac{1}{2}\int_{\tau_*} \eta(D_{ik} - \lambda G_{ik})^2 d\tau$$

$$R_\lambda \equiv 2R - \lambda\varphi^\cdot + \lambda^2\varphi = c_{\alpha\beta}(q_\alpha^\cdot - \lambda q_\alpha)(q_\beta^\cdot - \lambda q_\beta)$$

We will now require that $\lambda > 0$. Then, since D_λ, R_λ and T_λ are non-negative, Eqs.(7.1) yield the inequality

$$E_\lambda^\cdot \leqslant 2\lambda E_\lambda; \quad E_\lambda \equiv T_\lambda + \Pi_\lambda \tag{7.2}$$

integration of which gives

$$E_\lambda(t) \leqslant E_\lambda(0) \exp(2\lambda t) \tag{7.3}$$

We emphasize that inequality (7.3) holds for any solutions of problem (4.6)-(4.12) and for any positive values of the parameter λ. Moreover, we have not yet imposed any restrictions on the sign of the potential energy functional Π.

Since the quantity E_λ varies monotonically, it can be used as a Lyapunov functional.

8. The lower bound for the growth of perturbations. Let us assume now that inequality (4.14) holds. We choose the initial data (4.11), (4.12) from the set Q (4.14) so that $\Pi(0) < 0$. We assert that in that case one can always choose $E_\lambda(0) < 0$, after which it will follow from (7.3) that the perturbations increase at an exponential rate.

By (7.1), we have

$$E_\lambda(0) = E(0) + \lambda B(0) + \lambda^2 M(0), \quad 2B \equiv 2A - M^\cdot \tag{8.1}$$

Choose the initial data for the velocities so that $T(0) < |\Pi(0)|$, i.e., $E(0) < 0$. Then $E_\lambda(0)$ (8.1) is a second-degree polynomial in λ in which the coefficient $M(0)$ of λ^2 is positive and the free term $E(0)$ is negative. Therefore the conditions $\lambda > 0$, $E_\lambda(0) < 0$ are equivalent to the following admissible range of λ values:

$$0 < \lambda < \Lambda_1 \equiv -{}^1/_2 B/M + \sqrt{({}^1/_2 B/M)^2 - E/M} \tag{8.2}$$

Obviously, $\Lambda_1 > 0$ for any initial data with $E(0) < 0$. Under these conditions the value of Λ_1 is a lower bound for the increment. Indeed, if $\lambda = \Lambda_1 - \delta$ (where δ is any

number in the interval $0 < \delta < \Lambda_1$), inequality (7.3) is written as

$$E_{\Lambda_1-\delta}(t) \leqslant E_{\Lambda_1-\delta}(0) \exp[2(\Lambda_1 - \delta)t] \tag{8.3}$$

where $E_{\Lambda_1-\delta}(0) < 0$. It follows from the fact that T_λ is non-negative and the definition of Π_λ (7.1) that

$$E_\lambda(t) \equiv T_\lambda(t) + \Pi_\lambda(t) > \Pi(t)$$

and this relation, together with (8.3), yields

$$\Pi(t) < E_{\Lambda_1-\delta}(0) \exp[2(\Lambda_1 - \delta)t] \tag{8.4}$$

This inequality implies that the potential energy Π decreases without limit in its region of negative values. The drawback of (8.4), however, is that the definition of Π (4.5) involves not only deviations of q_α, N, but also the derivatives of the deviation $\nabla(N, N)$. At the same time, it is evident that dropping the term with $\nabla(N, N)$ from Π only strengthens inequality (8.4). Taking the moduli of the negative quantities and reversing the sign of the inequality, we infer from (8.4) that

$$I(t) > |I_1(t)| > |E_{\Lambda_1-\delta}(0)| \exp[2(\Lambda_1 - \delta)t] \tag{8.5}$$

$$2I_1 \equiv \left(\frac{\partial^2\Pi}{\partial q_\alpha \partial q_\beta}\right)_0 q_\alpha q_\beta + 2[\rho] q_\alpha \int_{\Gamma_0}\left(\frac{\partial\Pi_2}{\partial q_\alpha}\right)_0 N dS + \int_{\Gamma_0}\left\{[\rho]\left(\frac{\partial\Pi_2}{\partial\nu}\right)_0 + \sigma a\right\} N^2 dS,$$

$$I \equiv C\left(q_\alpha q_\alpha + \int_{\Gamma_0} N^2 dS\right), \quad C = \text{const} > 0$$

From (8.5) we conclude that the rest state (3.1) of the "body + liquid" system, when the potential energy (4.14) has no minimum, is unstable in the linear approximation in the mean square with respect to the deviations q_α, N. Under these conditions the increasing perturbations are bounded below by an exponential function with increment $\Lambda_1 - \delta$ (8.2), depending only on the initial data.

We now consider the solutions of (4.6)-(4.12) for which the initial field of velocities $\mathbf{v}$ and displacements ξ at each point satisfy the equations

$$\mathbf{v}(x_k, 0) = \xi_t(x_k, 0) = \lambda\xi(x_k, 0) \tag{8.6}$$

It follows from (7.1), (8.6) that $T_\lambda(0) = 0$, $E_\lambda(0) = \Pi_\lambda(0)$. The conditions $\lambda > 0$. $E_\lambda(0) < 0$ are then equivalent to restricting λ to the interval

$$0 < \lambda < \Lambda \tag{8.7}$$

$$\Lambda \equiv -\frac{A}{M} + \sqrt{\left(\frac{A}{M}\right)^2 - \frac{2\Pi}{M}} = \frac{-2\Pi}{A + \sqrt{A^2 - 2\Pi M}}$$

where the fact that Λ is positive is guaranteed by the choice of $\Pi(0) < 0$. Taking $\lambda = \Lambda - \delta$ with any number δ in the interval $0 < \delta < \Lambda$, we write (7.3) as

$$E_{\Lambda-\delta}(t) \leqslant \Pi_{\Lambda-\delta}(0) \exp[2(\Lambda - \delta)t]$$

whence we obtain the following bound, instead of (8.5):

$$I(t) > |I_1(t)| \geqslant |\Pi_{\Lambda-\delta}(0)| \exp[2(\Lambda - \delta)t] \tag{8.8}$$

It will be shown later that the lower bound for the growth of the perturbations with increment (8.7), (8.8) is in a certain sense the maximum admissible such bound.

9. The upper bound for the growth of perturbations. The fundamental inequality (7.3) also implies an upper bound for increasing perturbations. The idea underlying its derivation is to find a value of λ such that the functional (7.1) is positive definite for any displacement fields $\{\zeta, q_\alpha\}$. The functional E_λ will then also be positive definite, and, as a corollary, (7.3) will yield an upper bound for the perturbations.

The problem of determining the sign of Π_λ was considered in Sect.8. The conditions $\lambda > 0$, $\Pi_\lambda > 0$ for $\Pi < 0$ are equivalent to the inequality $\lambda > \Lambda$. Define Λ^+ by

$$\Lambda^+ = \sup_Q \Lambda \tag{9.1}$$

Then if $\lambda > \Lambda^+$ we have $\Pi_\lambda > 0$ for any displacements $\{\zeta, q_\alpha\} \in Q$. Thus, the functionals Π_λ and E_λ will be positive definite for all possible displacements $\{\zeta, q_\alpha\}$ if $\lambda = \Lambda^+ + \varepsilon$ for any real $\varepsilon > 0$.

Using the fact that $E_{\Lambda^+ + \varepsilon}$ is positive definite, we obtain the desired bound from (7.3):

$$E_{\Lambda^+ + \varepsilon}(t) \leqslant E_{\Lambda^+ + \varepsilon}(0) \exp[2(\Lambda^+ + \varepsilon)t]$$

and this inequality may be reduced, using the inequality $\Pi_{\Lambda^+} \geqslant 0$, to the intuitively clearer form

$$2T_{\Lambda^+ + \varepsilon}(t) + (2\varepsilon\Lambda^+ + \varepsilon^2) M(t) + \varepsilon(G + \varphi) \leqslant 2E_{\Lambda^+ + \varepsilon}(0) \exp[2(\Lambda^+ + \varepsilon) t] \tag{9.2}$$

Obviously, the mean square growth of the perturbations is bounded above by an exponential function with increment $\Lambda^+ + \varepsilon$.

10. Properties of the functional Λ. A necessary condition for the existence of a finite supremum Λ^+ as in (9.1) is that the function Λ (8.7) must be bounded above on the set Q (4.14). For simplicity, let us confine our attention to the case of a fixed vessel ($q_\alpha \equiv 0$) with a free surface ($\rho^- \equiv 0$). The desired bound is derived using the relations

$$\|\xi\|_{L_2(\Gamma_0)} \leqslant C_1 \|\xi\|_{H^1(\tau_0^+)}, \quad \|\xi\|_{H^1(\tau_0^+)} \leqslant C_2 G$$
$$\|\xi\|^2_{H^1(\tau)} \leqslant \int_\tau \left(\xi_i \xi_i + \frac{\partial \xi_i}{\partial x_k} \frac{\partial \xi_i}{\partial x_k}\right) d\tau, \quad \|\xi\|^2_{L_2(\Gamma)} \equiv \int_\Gamma \xi_i \xi_i d\tau \tag{10.1}$$

which hold for any fields $\xi(x_k) \in H^1(\tau_0^+)$, where C_1, C_2 are constants. Inequalities (10.1) are derived, e.g., in /12, p.138/ and /13, p.45/. The second inequality is a modification of Korn's inequality; an important point is that it will be valid even if $\xi = 0$ on only part of the boundary of τ_0^+. As to Λ (see (8.7)), we obtain, using (10.1) and (4.5),

$$\Lambda = \frac{-2\Pi}{G + \sqrt{G^2 - 2\Pi M}} \leqslant \frac{A_1}{G} \int_{\Gamma_0} |\xi|^2 dS \leqslant A_1 C_1 \|\xi\|_{H^1(\tau_0^+)}/G \leqslant A_1 C_1 C_2, \tag{10.2}$$
$$A_1 \equiv \sigma\rho^+ |\min_{\Gamma_0} a|$$

The bounds (10.2) for Λ and Λ^+ together with inequality (9.2), imply that the rate of growth of any perturbations of the class under consideration is bounded, and so the solution depends continuously on the initial data. For comparison, it should be mentioned that an ideal liquid with no surface tension does not have this property; there exist shortwave perturbations that increase as rapdily as desired (Rayleigh-Taylor instability /14/). The bound (10.2) is due to S.Ya. Belov.

Another interesting property of Λ_1, Λ (8.2), (8.7) is particularly evident when the generalized coordinates q_α ($\alpha = 1, \ldots, n$) include at least one cyclic coordinate $q_{\alpha'}$. By definition, the potential energy Π (4.5) of the system does not depend on this coordinate. At the same time, Λ depends on that coordinate through the quantity $2A \equiv G + \varphi$, $\varphi = c_{\alpha\beta} q_\alpha q_\beta$. Therefore, a displacement in the origin of the cyclic coordinate $q_{\alpha'} \to q_{\alpha'} + c$ will generally cause Λ to change. This property of the above bounds follows from the corresponding non-invariance of the generalized virial Eq.(5.3). The implication is that for every realization of the initial data (4.11) it makes sense to find the largest values of Λ_1, Λ, corresponding to $q_{\alpha'} = 0$.

The situation is considerably more complicated if allowance is made for the "analogues" of cyclic coordinates in a liquid, which always exist. It is obvious from (4.5) that the values of Π do not vary under all transformations of the functions $\zeta(x_k, t)$ which leave the field of normal displacements $N(x_k)$ on the surface Γ_0 invariant. For example, in (8.7) Π does not change, but G changes under a transformation $\zeta \to \zeta'$:

$$\zeta'(x_k, t) = \zeta(x_k, t) + \zeta_0(x_k)$$

with any field $\zeta_0(x_k)$ satisfying the relations

$$\operatorname{div} \zeta_0 = 0 \text{ in } \tau_0^+; \ \zeta_0 = 0 \text{ on } \partial\tau,$$
$$N \equiv \nu_0 \cdot \zeta_0 = 0 \text{ on } \Gamma_0$$

In actual fact, this transformation reduces to a redefinition of the equilibrium positions of the liquid particles. Here again, therefore, for every realization of the initial data (4.11) it makes sense to find the largest values of Λ_1, Λ corresponding to the given initial distortion of the boundary $N(x_k)$. It is precisely these maximum values that yield the best lower bound for the growth of a given perturbation. In accordance with the representation for (8.7), the variational problem reduces to finding the minimum of G given $N(x_k)$ on Γ and subject to the normalization condition $M \equiv 1$.

11. Models with moving a line of contact of three media. Let us return now to the general formulation of the problem as in Sects.2-4. The assumption that the curve γ is fixed (2.5) is contrary to experimental data /4-8/. However, allowing it to move, while at the same time assuming that the liquid satisfies the no slip condition, gives rise to a contradiction (infinite dissipation D_n (2.5) /3/). Considerable efforts have been made to overcome this contradiction /5-8/, but as yet no stable models (valid over the entire range of velocities) are available.

Below, in order to obtain bounds for the development of instability, we shall use three

models with a moving curve γ, in each of which the dynamic angle of contact θ is assumed to be constant (the Dupré-Young condition):

$$\cos\theta = (\sigma^- - \sigma^+)/\sigma \tag{11.1}$$

In the first model we adopt Navier's slippage condition /5, 6/, according to which the velocity of the liquid at the surface $\partial\tau$ is proportional to the shear stress $\mu_i \equiv \sigma_{ik}n_k - \sigma_{km}n_k n_m n_i$:

$$\boldsymbol{\mu} = -\varkappa\mathbf{u},\ \varkappa = \text{const} > 0;\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\tau \tag{11.2}$$

Two other modifications of the boundary conditions are models with a given slippage interval /5, 6/, according to which the rigid surface $\partial\tau$ is divided into two parts: in a strip $\partial\tau_\gamma(\beta)$ (of width 2β with middle curve γ, $\beta = \text{const}$) either the Navier slippage condition holds or the shearing stresses vanish; in the other part $\partial\tau_1 \equiv \partial\tau \setminus \partial\tau_\gamma(\beta)$ there is no slip:

$$\mathbf{u} = 0 \text{ on } \partial\tau;\ \mathbf{u}\cdot\mathbf{n} = 0,\ \boldsymbol{\mu} = -\varkappa\mathbf{u} \text{ on } \partial\tau_\gamma \tag{11.3}$$

$$\mathbf{u} = 0 \text{ on } \partial\tau_1;\ \mathbf{u}\cdot\mathbf{n} = 0,\ \boldsymbol{\mu} = 0 \text{ on } \partial\tau_\gamma \tag{11.4}$$

where $\varkappa = \varkappa^\pm$ in (11.2), (11.3) for the liquids with densities $\rho^\pm$. Conditions (11.4), a special case of (11.3) with $\varkappa = 0$, is essentially the same as the developed in /15/.

Thus, Eqs.(2.1)-(2.3), (11.1), plus one of conditions (11.2)-(11.4), furnish three more possible formulations of the exact problem. Relative to solutions of any of these problems the energy equality (2.6) is replaced by the relation

$$E_n^{\cdot} = -D_n - D_n' - q_\alpha^{\cdot}\partial R_n/\partial q_\alpha^{\cdot}, \quad D_n' \equiv \int_{\partial\tau_\gamma} \varkappa u_i u_i dS$$

in which D_n' is the dissipation of energy due to surface friction. In model (11.2) $\partial\tau_\gamma = \partial\tau$; in models (11.4) or (2.4), (2.5) $D_n' = 0$.

The linearized versions of these three models include Eqs.(4.6)-(4.9), to which one must add the linearized Dupré-Young condition (11.1) /4/:

$$\partial N/\partial e + \chi N = 0 \text{ on } \gamma_0 \tag{11.5}$$

where γ_0, χ, e are defined in (3.2).

Conditions (11.2)-(11.4) retain the same form when the problem is linearized, and together with (4.6)-(4.9), (11.5) give three new formulations. For the solutions of any of these problems (4.13) is replaced by

$$E^{\cdot} = -D - D' - 2R, \quad D' \equiv \int_{\partial\tau_{\gamma 0}} \varkappa u_i u_i dS \tag{11.6}$$

The notation is the same as in (4.13), except that in Π (4.5) the last term (the integral over γ_0) does not vanish.

The generalized virial Eq.(6.3) maintains the same form in these models, except a new term appears in the functional X:

$$X = M^{\cdot} + G + \varphi + V, \quad V \equiv \int_{\partial\tau_{\gamma 0}} \varkappa\zeta_i\zeta_i dS \tag{11.7}$$

As a consequence of (11.6) and (11.7), new terms also appear in (7.1) and the definition of Π_λ:

$$d/dt\,(T_\lambda + \Pi_\lambda) = -2\lambda\,(T_\lambda - \Pi_\lambda) - D_\lambda - R_\lambda - D_\lambda' \tag{11.8}$$

$$D_\lambda' \equiv D' - \lambda V^{\cdot} + \lambda^2 V = \int_{\partial\tau_{\gamma 0}} \varkappa\,(\zeta_t - \lambda\xi)\,dS$$

$$2\Pi_\lambda \equiv 2\Pi + 2\lambda A + \lambda^2 M,\ 2A \equiv G + \varphi + V$$

At the same time, since D_λ' is non-negative, the fundamental inequalities (7.2) and (7.3) remain unchanged.

In accordance with (11.8), only the definition of A an estimates (8.2), (8.3),(8.5), (8.8) and (9.2) changes, and I (8.6) must also include an integral along γ_0:

$$I \equiv C\left(q_\alpha q_\alpha + \int_{\Gamma_*} N^2 dS + \int_{\gamma_0} N^2 dl\right)$$

Thus, the results of Sects.8 and 9 remain valid even when the no-slip conditions for the

liquid at the rigid boundary $\partial\tau$ are relaxed.

Remarks. 1°. The fundamental inequalities (8.5), (8.8) and (9.2) have the nature of a priori estimates,since the appropriate existence theorems for the solutions have not been proved.

2°. The bound (8.8) for the increments when there is no dissipation (an ideal liquid) was obtained previously by other methods /11/.

3°. The condition adopted in Sect.11, according to which the dynamic angle of contact is constant, is one of the models used in /3-8/. An analysis of this problem is beyond the scope of this paper. Nevertheless, the generality of our account enables us to hope that the hydrodynamic analogue of the inversion of Lagrange's theorem should be valid for other boundary conditions on $\partial\tau$, provided that they admit of linearization and are compatible with the non-increasing energy condition.

4°. One of the general qualitative conclusions from the form of the bounds (8.5), (8.8) is that the presence of a perturbation with negative potential energy (4.14) will cause instability to develop for as large (but finite) values of the dissipative coefficients $\eta^{\pm}$, $\varkappa^{\pm}, c_{\alpha\beta}$ as desired. In other words, an increase in the dissipative factors will not stabilize the system.

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